UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Monday 3 June 2019 9.00 to 11.00 am

PHYSICS (7)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (7)

QUANTUM CONDENSED MATTER PHYSICS

Candidates offering this paper should attempt a total of ***five*** *questions:* ***three*** *questions from Section A and* ***two*** *questions from Section B.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains ***five*** *sides, including this coversheet, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
2 × 20 Page Answer Book	Mathematical Formulae handbook
Metric graph paper	Approved calculator allowed
Rough workpad	
Yellow master coversheet	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A

*Attempt **all** questions in this Section. Answers should be concise and relevant formulae may be assumed without proof.*

1 State the form of the wavefunction for an electron in a crystal. What is meant by the crystal momentum of the electron and how does it differ from the true momentum of the electron? [4]

2 Write down the form of the resulting potential $V(r)$ at a distance r from an extra fixed charge Q inserted into a conductor. Sketch, on the same graph, $V(r)$ and the bare potential from the same charge Q. Explain how the length scale of the variation of the potential $V(r)$ depends on a property of the conductor. [4]

3 Two sheets of monolayer graphene are laid flat, one on top of the other, with a small angle between the corresponding in-plane crystal axes in the two layers. This produces Moiré fringes, with periodicity $50a$, where $a = 0.25$ nm is the period of the graphene lattice. Consider, for simplicity, just a one-dimensional chain with these periodicities, and assume a *linear* dispersion relation with group velocity $v = 10^6$ ms^{-1}. Assuming that the only perturbation felt by conduction electrons is from the weak lattice and superlattice potentials, estimate the period in energy of the resulting energy spectrum. [4]

SECTION B
*Attempt **two** questions from this section*

4 Discuss the behaviour of a free-electron gas in the presence of a weak periodic potential, and indicate why this leads to the appearance of energy gaps at the Brillouin-zone boundaries. [5]

A one-dimensional lattice, with period a, has a weak lattice potential $V(x)$ given by

$$V(x) = -2V_0 \cos gx \, .$$

Here, $g \equiv 2\pi/a$ and $V_0 \ll \epsilon_{g/2}$, where ϵ_k denotes the free-electron energy at wavevector k. By using an approximate wavefunction of the form

$$\psi(x) = A\mathrm{e}^{\mathrm{i}k_+ x} + B\mathrm{e}^{\mathrm{i}k_- x},$$

where $k_\pm = q \pm g/2$ (with $q \ll g/2$), show that the energy $E(q)$ is given approximately by

$$E(q) = \frac{1}{2}\left[\epsilon_{q+g/2} + \epsilon_{q-g/2} \pm \left((\epsilon_{q+g/2} - \epsilon_{q-g/2})^2 + 4V_0^2\right)^{1/2}\right] \, .$$

[4]

By expanding $E(q)$ in powers of q, show that the effective masses m^* at the Brillouin-zone boundary are given by

$$\frac{m^*}{m} = \pm\frac{V_0}{2\epsilon_{g/2}}$$

where m is the free-electron mass. [4]

Sketch typical phonon dispersion curves for acoustic and optical phonons. Use this diagram to explain how the range of available phonon momenta depends on temperature in the low-temperature regime.

A particular simple-cubic crystal has a lattice constant $a = 0.5$ nm, sound velocity $v_\mathrm{s} = 3\,\mathrm{km\,s^{-1}}$, and Fermi wavevector $k_\mathrm{F} = 0.7\pi/a$. Estimate the temperature above which the low-temperature electrical conductivity might be expected to decrease strongly, explaining your reasoning. [4]

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5 For a pure metal in a magnetic field B, the flux enclosed by the quantised motion of a charge $-e$ in the plane perpendicular to the magnetic field is

$$\Phi_n = \left(n + \frac{1}{2}\right)\frac{h}{e},$$

where n is some integer and h is Planck's constant. Explain this phenomenon and, by considering the Lorentz force, find the area enclosed by the corresponding motion in k-space. [6]

Explain why this result leads to oscillations in various properties of the material, describing one of the observable effects. [3]

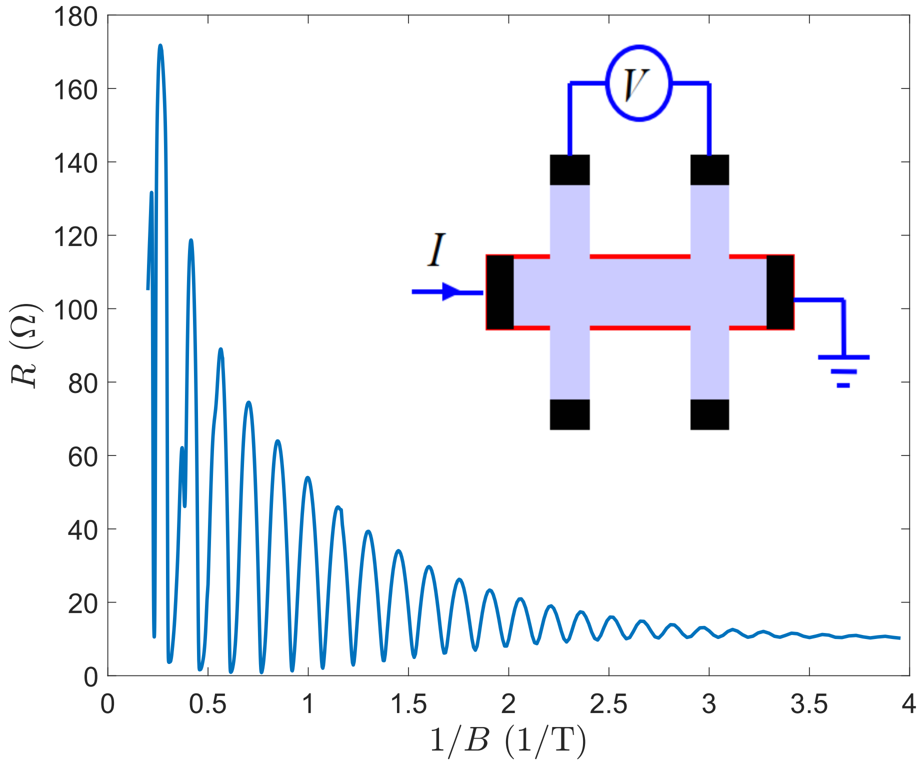

Figure 1: Four-terminal resistance of a 2D system.

A 'Hall bar' made from a *different* material containing a high-mobility *two*-dimensional electron system has the four-terminal resistance V/I shown in Figure 1 as a function of $1/B$. Explain the origin of the peaks and why some of them split. [3]

Using the fact that each state in a given 'level' corresponds to a unit of flux h/e, show that the number of occupied states per unit area is $eB\nu/h$, defining what the number ν represents. Find the 2D carrier density for this sample. [3]

Discuss why scattering (with characteristic time τ) and finite temperature T reduce the amplitude of the resistance oscillations seen at large $1/B$ in the figure. Sketch the Hall resistance *vs* B in such samples and describe what is seen at higher fields in the best samples. [4]

6 Show that the magnetisation of a paramagnetic insulator with n localised magnetic ions per unit volume, each with total angular momentum $J = ½$, is

$$M = n\mu_B \tanh \frac{\mu_B B}{k_B T}$$

at temperature T in a magnetic field B. [4]

Show how this leads to Curie's law $\chi = C/T$. [2]

Consider a ferromagnetic insulator where the effect of the surrounding ions is approximated in the above model by an effective magnetic field of the form $\boldsymbol{B}_E = \mu_0 \boldsymbol{H}_E = \mu_0 \lambda \boldsymbol{M}$ (with $\lambda > 0$). Show that, when there is no applied magnetic field,

$$m = \tanh \frac{m}{t} \,,$$

where $m = |\boldsymbol{M}|/n\mu_B$ and t is to be found. By sketching functions on a graph, find when this has solutions with $m > 0$. [3]

Show that, for non-zero $m \ll 1$, $m \approx t\sqrt{3(1-t)}$ and sketch m against t for $-\infty < 1 - t \ll 1$. [4]

Explain the physical mechanisms giving rise to this effective field ($\boldsymbol{H}_E$), how it can be represented by the Heisenberg Hamiltonian, and how this can describe ferromagnetism or antiferromagnetism in various types of insulator. [6]

$\left[\textit{Note that } \tanh x = x - \frac{1}{3}x^3 + O(x^5) \,. \right]$

END OF PAPER