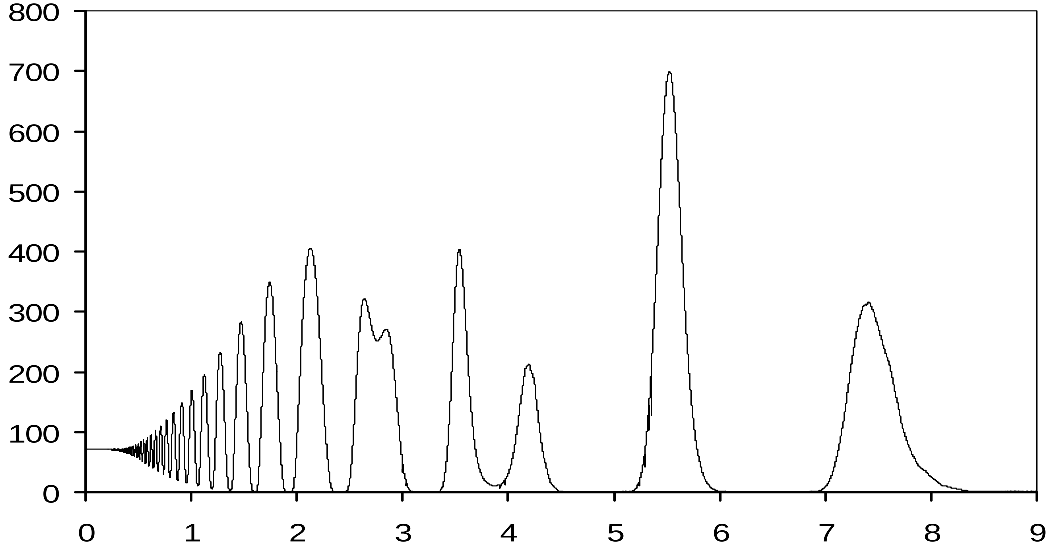
# Advanced Quantum Physics

## Lecture 13




**David Ritchie**

**www.sp.phy.cam.ac.uk/~dar11/pdf**

# Section 3:Relativity and Magnetic fields


3.1 Quantum mechanics and magnetic fields

3.2 Relativistic quantum mechanics

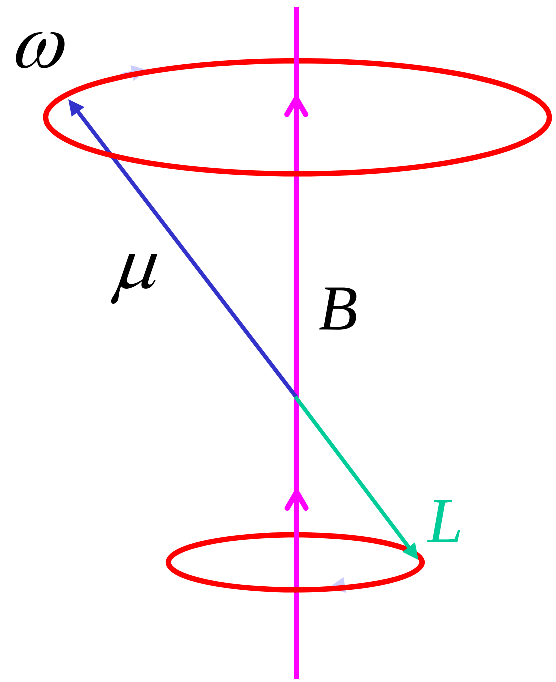
3.3 Particle in a uniform magnetic field

✦ 3.4 Landau levels

# Larmor Precession (1)

- From the last lecture, in a uniform field $\mathbf{B} = (0, 0, B_z)$
In the *symmetric* gauge: $\mathbf{A} = \frac{1}{2} B_z(-y, x, 0)$ .

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m_e} + \frac{e}{2m}(\hat{\mathbf{L}} + g\hat{\mathbf{S}}) \cdot \mathbf{B} + \frac{e^2}{8m}[B^2 r^2 - (\mathbf{B} \cdot \mathbf{r})^2]$$




- There is a magnetic moment associated with $\hat{\mathbf{L}}$:

$$\hat{\boldsymbol{\mu}}_{\mathbf{L}} = -\frac{e}{2m}\hat{\mathbf{L}}$$

- Classically we expect precession about $\mathbf{B}$
- also happens in Quantum Mechanics.

- Now:

$$\frac{d}{dt}\langle \hat{O} \rangle = \frac{i}{\hbar}\langle [\hat{H}, \hat{O}] \rangle \qquad \text{hence} \qquad \frac{d}{dt}\langle \hat{L}_x \rangle = \frac{i}{\hbar}\langle [\hat{H}, \hat{L}_x] \rangle$$

- If $\mathbf{B}$ is along the z-axis, neglecting spin and the $B^2$ term in the Hamiltonian.

- Then:

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m_e} + \frac{e}{2m}\hat{L}_z B_z$$

# Larmor Precession (2)

- From the previous slide:

$$\frac{d}{dt}\langle \hat{L}_x \rangle = \frac{i}{\hbar}\langle [\hat{H}, \hat{L}_x] \rangle \qquad \hat{H} = \frac{\hat{\mathbf{p}}^2}{2m_e} + \frac{e}{2m}\hat{L}_z B_z$$

- Now $\left[\hat{\mathbf{p}}^2, \hat{L}_x\right] = 0$ hence: $$\frac{d}{dt}\langle \hat{L}_x \rangle = \frac{i}{\hbar}\frac{eB_z}{2m}\langle [\hat{L}_z, \hat{L}_x] \rangle$$

- Since $\left[\hat{L}_z, \hat{L}_x\right] = i\hbar \hat{L}_y$ then $$\frac{d}{dt}\langle \hat{L}_x \rangle = -\frac{eB_z}{2m}\langle \hat{L}_y \rangle$$

- Similarly $\frac{d}{dt}\langle \hat{L}_y \rangle = \frac{eB_z}{2m}\langle \hat{L}_x \rangle$ - the solutions of these equations are

$$\langle \hat{L}_x \rangle = A\cos(\omega_L t + \alpha) \quad ; \quad \langle \hat{L}_y \rangle = A\sin(\omega_L t + \alpha)$$

where $\omega_L = eB_z/2m$ is called the *Larmor* frequency. These equations represent precession of $\mathbf{L}$ around $\mathbf{B}$ at a frequency $\omega_L$.

- This is the same result as obtained classically

# Landau Levels (1)

- The Hamiltonian for an electron moving in a magnetic field is:

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m_e} + \frac{e}{2m}(\hat{\mathbf{L}} + g\hat{\mathbf{S}})\cdot\mathbf{B} + \frac{e^2}{8m}[B^2 r^2 - (\mathbf{B}\cdot\mathbf{r})^2]$$

- Consider the motion of an electron in eigenstates of $\hat{L}_z$ and $\hat{S}_z$, moving in a uniform field $B_z$ which is not small – we can't neglect the $B^2$ term. The Hamiltonian for the electron is,

$$\hat{H} = \frac{\hat{p}_z^2}{2m} + \left(m_\ell + g m_s\right)\mu_B B_z + \left(\frac{\hat{p}_x^2}{2m} + \frac{e^2 B_z^2}{8m}x^2\right) + \left(\frac{\hat{p}_y^2}{2m} + \frac{e^2 B_z^2}{8m}y^2\right)$$

- The first term represents free motion in the z-direction.

- The second term is due to orbital angular momentum and spin with quantum numbers $m_\ell$ and $m_s = \pm\frac{1}{2}$ respectively.

- The last two terms are of the form of harmonic oscillator Hamiltonians in the x- and y-direction. $\frac{1}{2}m\omega^2 = e^2 B^2/8m$

Hence the electron executes simple harmonic motion in the x-y plane with frequency $\omega$ where: $\omega = \frac{eB}{2m} = \frac{\mu_B B}{\hbar} \equiv \omega_L$ The Larmor frequency (again).

# Landau Levels (2)

•From the last slide, the electron oscillates in the x-y plane with frequency:

$$\omega = \frac{eB}{2m} = \frac{\mu_B B}{\hbar} \equiv \omega_L$$

•The x and y oscillators will each have energy:

$$\hbar\omega_L(n_{x,y} + \tfrac{1}{2}) \quad \text{where} \quad n_{x,y} = 0,1,2,\ldots$$

and adding in the contributions from orbital and spin angular momentum, the overall energy levels will be of the form;

$$E = \frac{p_z^2}{2m} + (m_\ell + n_x + n_y + 1)\hbar\omega_L + g m_s \hbar\omega_L$$

where $p_z$ can take any value.

•In this case the scalar potential $\phi = 0$ and since the Hamiltonian can then be written as a square $\hat{H} = \frac{1}{2m}\left(\hat{\mathbf{p}} - q\mathbf{A}\right)^2$ the energy eigenvalues, $E$ , must be positive or zero.

•This condition must hold for $p_z = 0$ and $m_s = -1/2$ and given $g \approx 2$ then:

$$m_\ell + n_x + n_y \geq 0$$

# Landau Levels (3)

- The harmonic oscillator part of the Hamiltonian:

$$\hat{H}_{xy} = \left( \frac{\hat{p}_x^2}{2m} + \frac{e^2 B_z^2}{8m} x^2 \right) + \left( \frac{\hat{p}_y^2}{2m} + \frac{e^2 B_z^2}{8m} y^2 \right)$$

Is invariant under the reflections $x \to -x, \quad y \to -y$ hence the wavefunctions are eigenstates of the parity operator and therefore even or odd under reflection.

- From 1D harmonic oscillator wavefunctions the 2D wavefunctions are even if $n_x + n_y$ is even and odd if $n_x + n_y$ is odd.
- The angular variation of the wavefunction is of the form: $e^{im_\ell \phi}$ which is even when $m_\ell$ is even and odd when $m_\ell$ is odd.
- Hence both $n_x + n_y$ and $m_\ell$ should be even or both should be odd. This implies $m_\ell + n_x + n_y$ should be even and we can write: $m_\ell + n_x + n_y = 2n$
- So we have:

$$E = \frac{p_z^2}{2m} + (2n+1)\hbar\omega_L + g m_s \hbar\omega_L \,, \quad n = 0,1,2,\ldots, \quad m_s = \pm\tfrac{1}{2}.$$

# Landau Levels (4)

•From the last slide:

$$E = \frac{p_z^2}{2m} + (2n+1)\hbar\omega_L + g m_s \hbar\omega_L \ , \quad n = 0,1,2,\ldots, \quad m_s = \pm\tfrac{1}{2}.$$

•The second term gives a series of states known as *Landau* energy levels..

•These *Landau Levels* are further split by the effects of electron spin in the third term.

•If you choose a different gauge you may get different eigenfunctions but will get the same energy levels.

•For example with $\mathbf{B} = (0, 0, B)$ in the *Landau* gauge: $\mathbf{A} = (0, Bx, 0),\ \phi = 0$ gives plane wave eigenfunctions in the y-direction with harmonic oscillator eigenfunctions in the x-direction (see example). However this gauge will give the *same* energy levels as described above in the *symmetric* gauge.

•The form of the eigenfunctions in the *Landau* gauge can be reconciled with the solution described above in the *symmetric* gauge because this is a highly degenerate system – the overall wavefunction can be a sum of many degenerate levels.

# Two-dimensional electron gas (1)

•Landau levels are important in the physics of a 2D electron gas

•A layer of electrons is confined in z-direction by a quantum well, the wavefunction in that direction is a standing wave, travelling waves in the x- and y-directions.

•Electrons fill states up to the Fermi energy.

•In 2D the density of states is calculated:

•Consider a lattice of points in k-space,

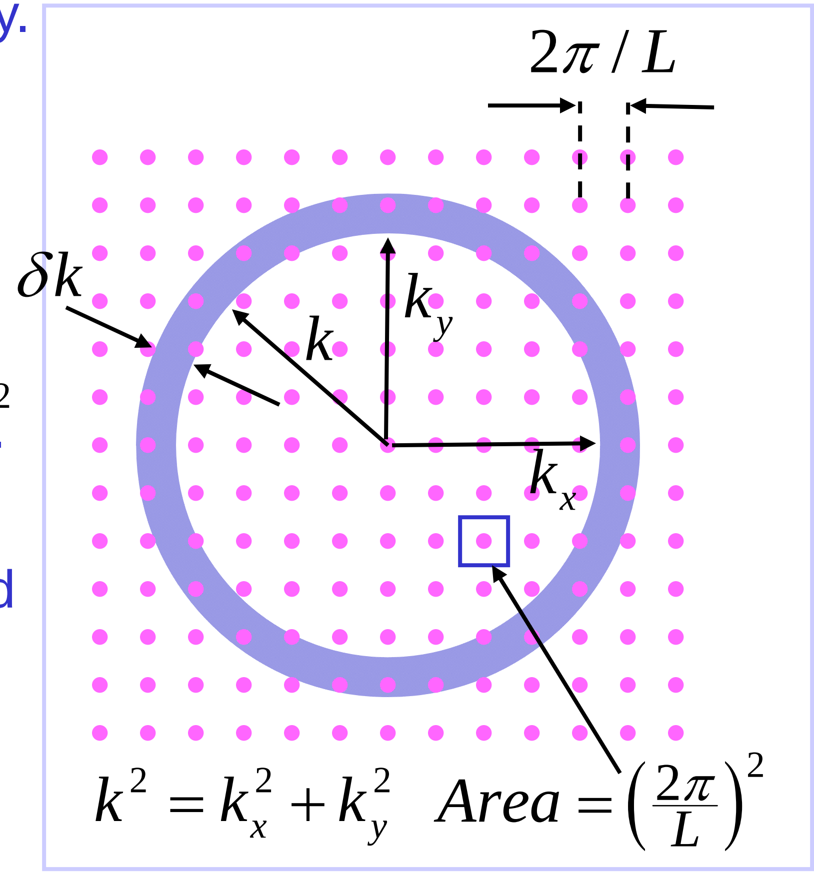
•2D gas $L \times L$ in size.

•Spacing between points is $2\pi / L$,area in k-space occupied by each mode: $(2\pi / L)^2$.

• If states filled to $k^2 = k_x^2 + k_y^2$, area in k-space of occupied states between $k$ and $k + \delta k$ is $2\pi k \delta k$ .

•No. of occupied states in this region:

$$\delta n = \frac{2\pi k \delta k}{(2\pi / L)^2} = \frac{k \delta k L^2}{2\pi}$$



# Two-dimensional electron gas (2)

$$\omega_L = \frac{eB}{2m}$$

•From the last slide no. of states between $k$ and $k+\delta k$

$$\delta n = \frac{2\pi k\delta k}{(2\pi/L)^2} = \frac{k\delta k L^2}{2\pi}$$ hence $$\frac{dn}{dE} = \frac{kL^2}{2\pi}\frac{\delta k}{\delta E}$$

•For a free electron: $E = \hbar^2 k^2/2m$ so $$\frac{\partial k}{\partial E} = \frac{m}{\hbar^2 k} \Rightarrow \frac{dn}{dE} = \frac{L^2}{2\pi}\frac{m}{\hbar^2}$$

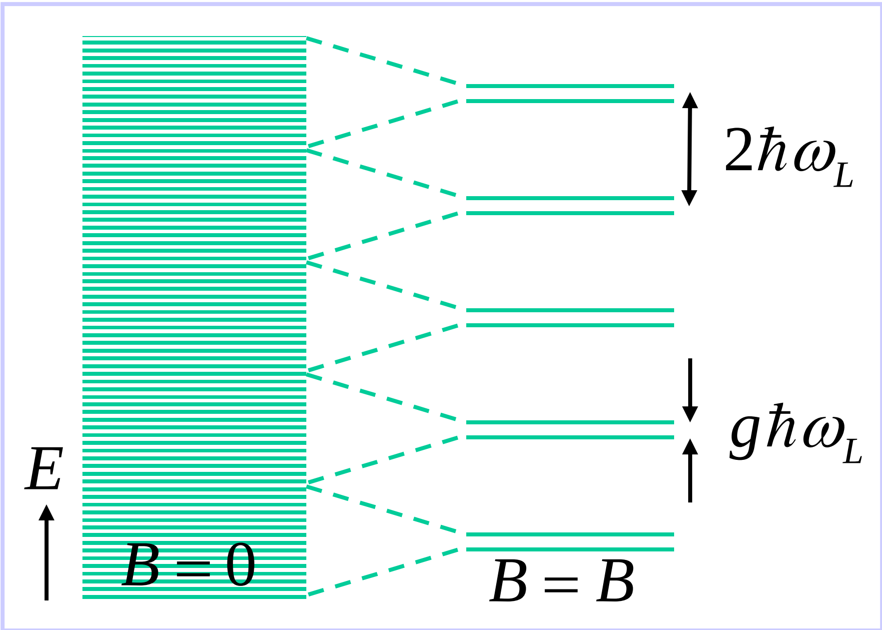
•Hence per unit area, with a factor of 2 for spin, the density of states is: $$g(E) = \frac{2}{L^2}\frac{\partial n}{\partial E} = \frac{m}{\pi\hbar^2}$$ (independent of $E$).

•Application of a magnetic field in the z-direction.

•Constant density of states splits into pairs of Landau levels.

•Difference in average energy of adjacent pairs: $\Delta E = 2\hbar\omega_L$

•Pair splitting due to spin-determined by electron g-factor: $\Delta E_s = g\hbar\omega_L$



# Two-dimensional electron gas (3)

- As magnetic field increases , energy splitting between Landau levels increases:

$$\Delta E = 2\hbar\omega_L = \frac{\hbar eB}{m}, \qquad (\omega_L = eB/2m)$$

- As field increases highest Landau levels become depopulated one-by-one and the electrons distributed to other levels.
- If occupation of a Landau level per unit area is: $n_L$
- Taking into account spin degeneracy the average density of states in presence of a field is $2n_L / 2\hbar\omega_L = n_L / \hbar\omega_L$
- Equating this to density of states at $B = 0$: $g(E) = m/\pi\hbar^2$

$$n_L = \frac{m\omega_L}{\pi\hbar} = \frac{eB}{2\pi\hbar}$$

- If there are $\upsilon$ filled Landau levels at a field $B_1$ the total density of electrons per unit area is given by:

$$n_e = \frac{\upsilon e B_1}{2\pi\hbar}$$

# Two-dimensional electron gas (4)

- Suppose there are $\upsilon$ occupied Landau levels: $n_e = \upsilon e B_1 / 2\pi\hbar$
- If the field is increased from $B_1$ to $B_2$ and the highest Landau level is depopulated.
- The electrons are re-distributed among $\upsilon - 1$ levels.
- Hence:

$$n_e = \upsilon \frac{eB_1}{2\pi\hbar} = (\upsilon - 1)\frac{eB_2}{2\pi\hbar}$$

- On eliminating $\upsilon$:

$$n_e = \frac{e}{2\pi\hbar}\left(\frac{1}{B_1} - \frac{1}{B_2}\right)^{-1}$$

- This means that the depopulation of the Landau levels is periodic in $\frac{1}{B}$
- At low temperatures where $KT \ll \hbar\omega$ the depopulation of the Landau levels is seen in the resistance of a high mobility 2D electron gas.
- Fermi level between Landau levels - behaves like an insulator
- Fermi level in a Landau level – behaves like a conductor.
- Resistance oscillates with $\frac{1}{B}$ - Shubnikov-de-Hass effect.

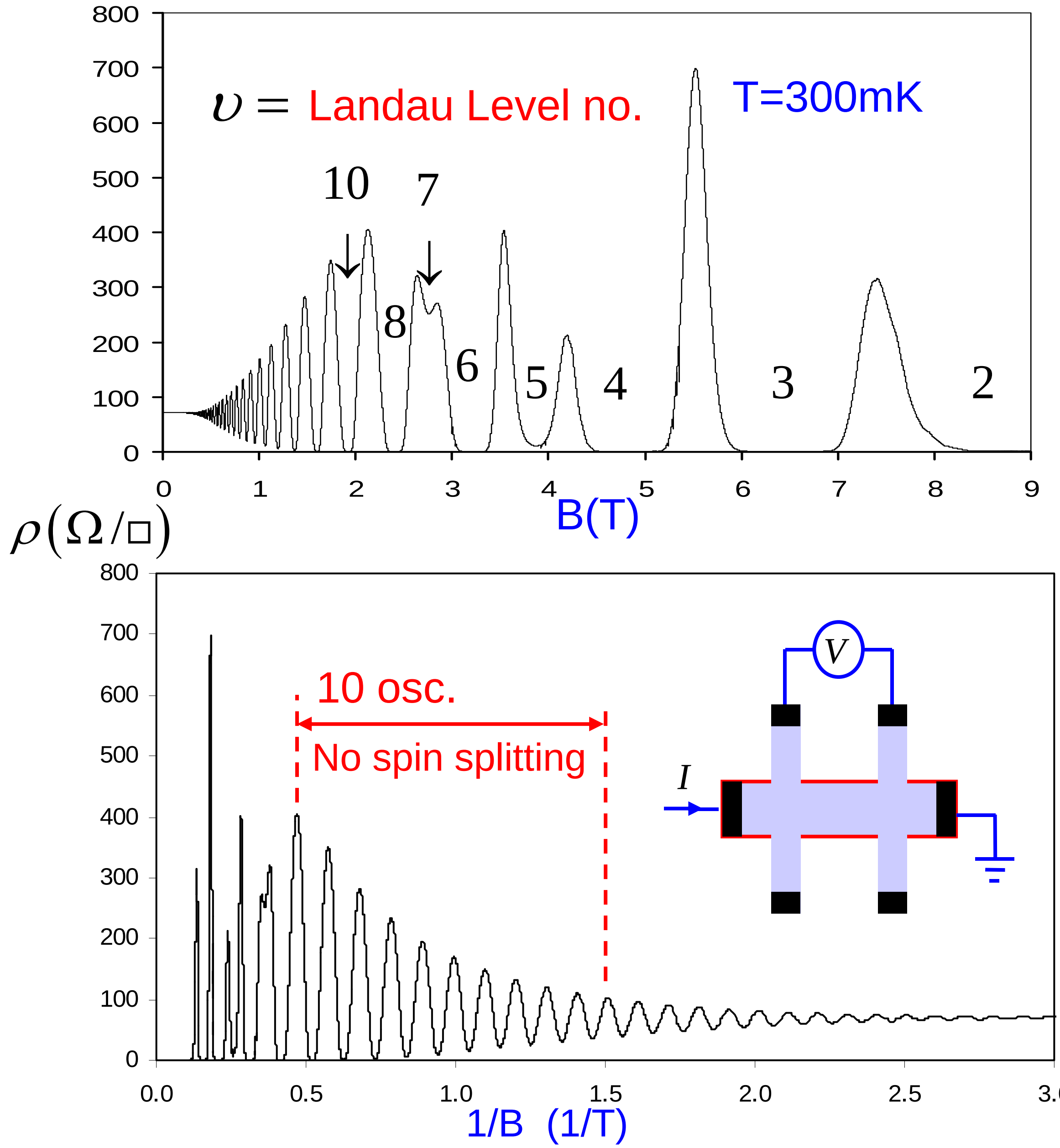
# Shubnikov-de-Hass effect

- Oscillations in resistance of a high mobility 2D electron gas.
- Periodic in 1/B
- Calculate electron density
- 10 oscillations in $1.05T^{-1}$
- Each oscillation is two levels - no spin splitting (yet)

$$\frac{1}{B_1}-\frac{1}{B_2}=\frac{0.105}{2}T^{-1}$$

$$n_e=\frac{e}{2\pi\hbar}\left(\frac{1}{B_1}-\frac{1}{B_2}\right)^{-1}$$

$$\Rightarrow n_e=4.60\times10^{15}m^{-2}$$



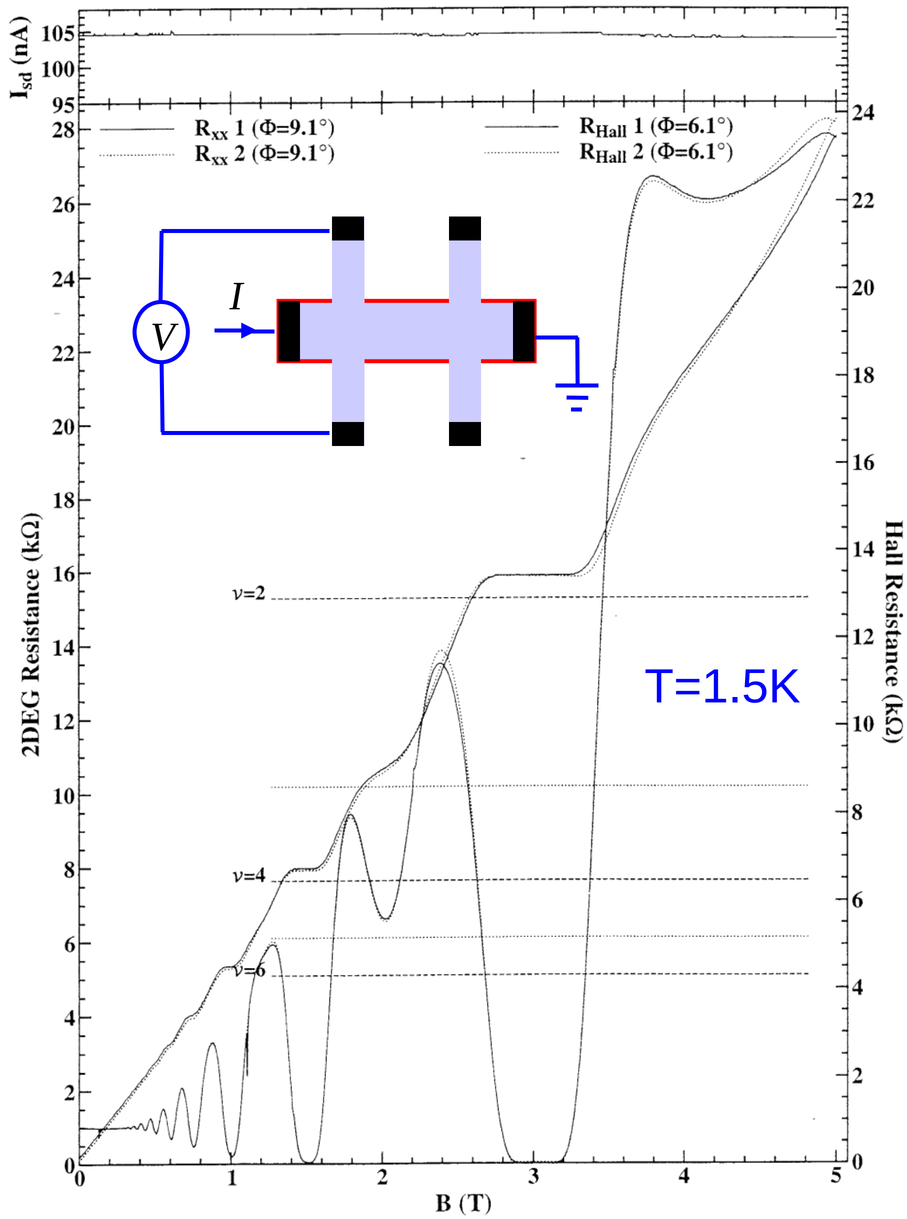
# Quantum Hall effect

- Measure voltage perpendicular to current - Hall effect : $R_h = \frac{V}{IB} = \frac{1}{n_e e}$
- $n_e$ density/unit area
- In high mobility 2D electron gas Hall voltage deviates from straight line forming plateaux when there are full Landau levels - where:

$$R_h = \frac{V}{IB} = \frac{2\pi\hbar}{\upsilon e^2} = \frac{25812.807}{\upsilon}\Omega$$

with $\upsilon$ being the no. of filled Landau levels at the plateaux.

- When $R_h$ on plateau resistance $\rightarrow 0$
- Since 1990 used as standard of resistance (5 parts in $10^8$).
- Effect very insensitive to sample disorder, astonishing accuracy for solid state experiment, Nobel prize 1985 awarded to von Klitzing.

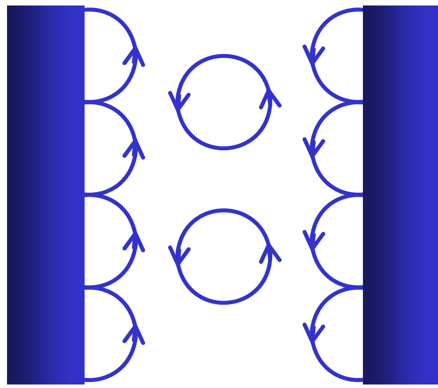


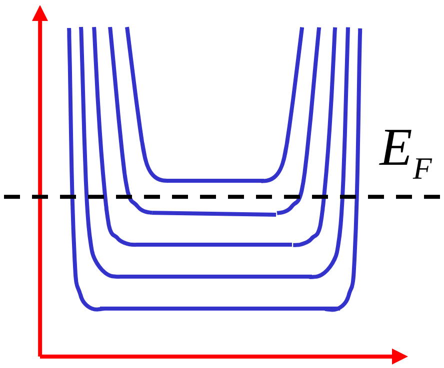

von Klitzing, Dorda and Pepper Phys Rev Lett. **45**, 494 (1980)

# Quantum Hall effect (2)

- Best explanation concerns 'edge states'

- In a high magnetic field classically electrons will move in circles and 'skip' along the edge of the sample.



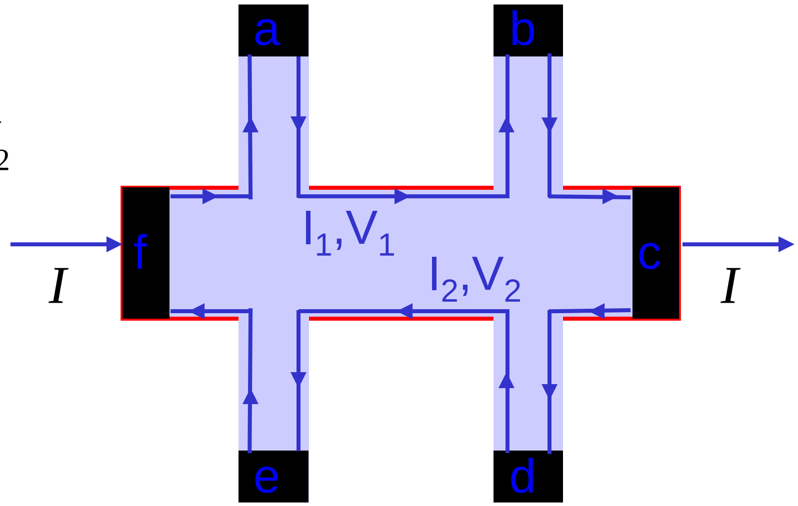




- In Landau Level picture – levels rise at edges of samples forming edge states.

- With $\upsilon$ full Landau levels only $\upsilon$ edge states contribute to conduction.

- Can regard edge states as 1D conductors :

$$g = \frac{\partial I}{\partial V} = \upsilon \frac{e^2}{2\pi\hbar} \Rightarrow I_1 = \upsilon \frac{e^2}{2\pi\hbar} V_1, \quad I_2 = \upsilon \frac{e^2}{2\pi\hbar} V_2$$




- With net current   flowing into sample:

$$\Rightarrow I = I_1 - I_2 = \upsilon \frac{e^2}{2\pi\hbar}(V_1 - V_2)$$

$$\Rightarrow \frac{V_1 - V_2}{I_1 - I_2} = \frac{V_a - V_e}{I} = \frac{2\pi\hbar}{\upsilon e^2} = \frac{25812.807}{\upsilon}\Omega$$

- With this picture quantum Hall plateaux very narrow - to observe them we must introduce disorder and localize some electrons!

# Fractional Quantum Hall Effect

**Willett et al PRL 59,1776 (1987)**

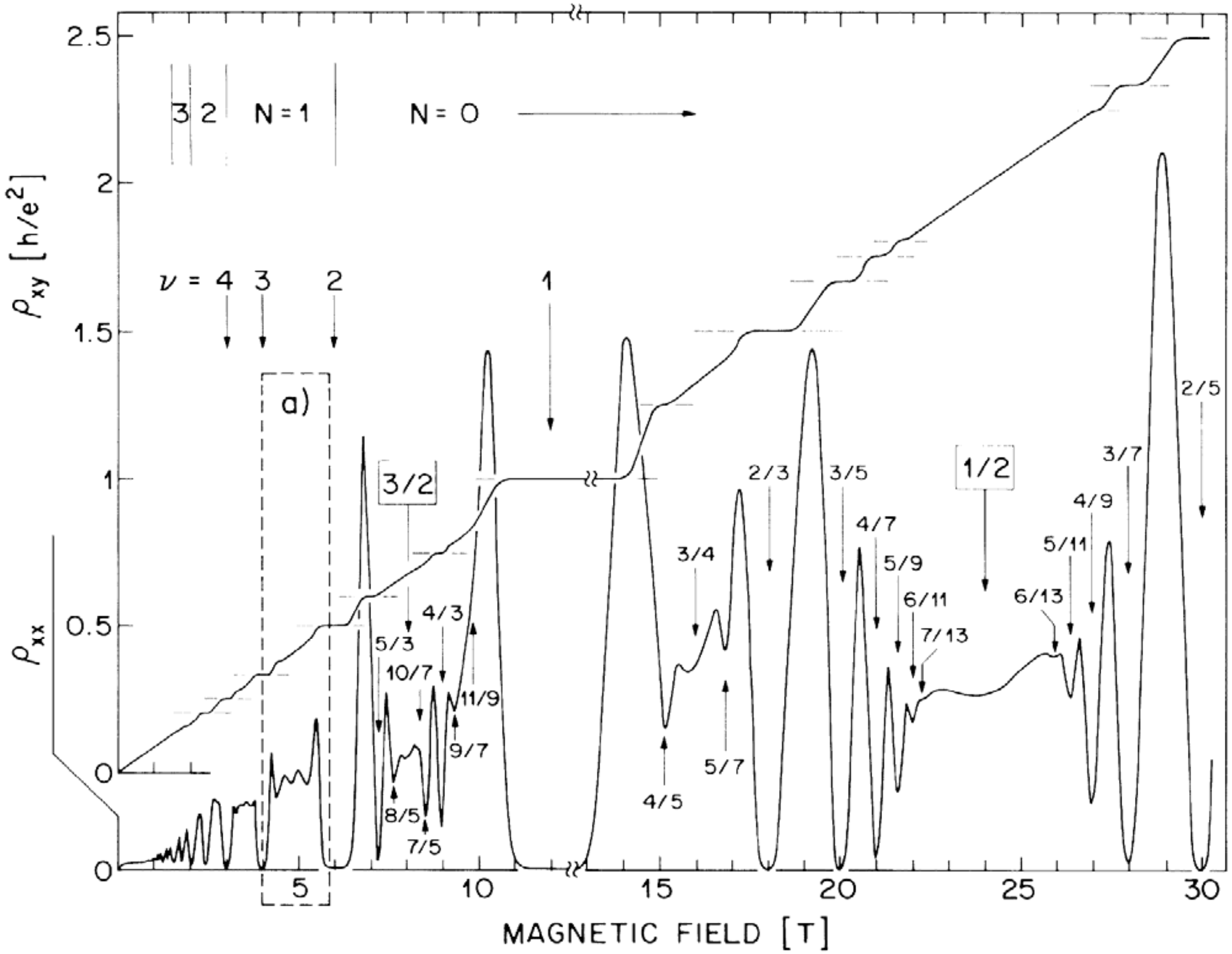
Electron mobility

$1.6\text{x}10^{6}\text{cm}^{2}\text{V}^{-1}\text{s}^{-1}$

Electron density

$3.0\text{x}10^{11}\text{cm}^{-2}$

Temperature
150mK




Not examinable

# Lecture 13 - Summary

•Larmor precession – motion of a magnetic moment in a magnetic field.

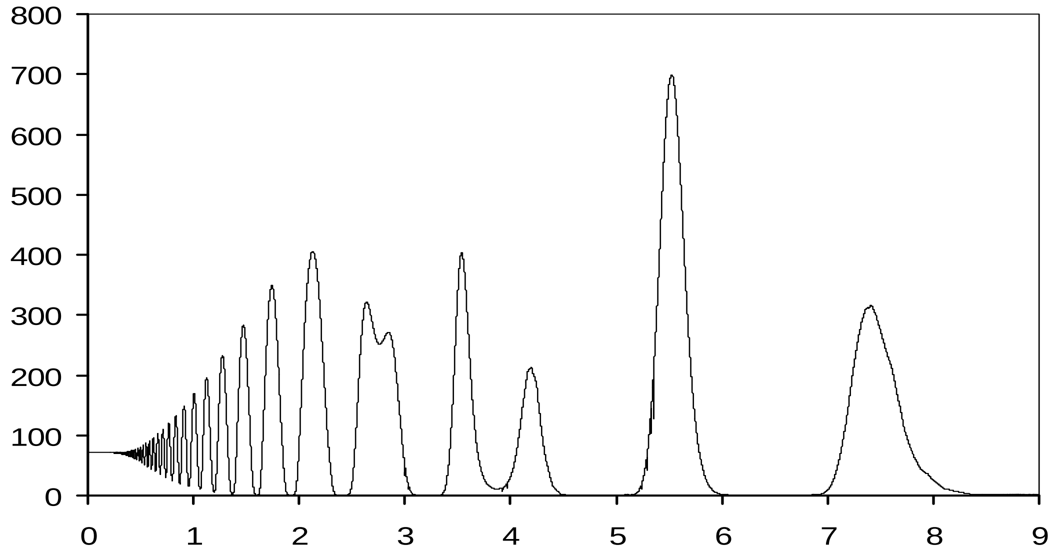
•Landau levels – harmonic oscillator potential in two-dimensions, calculation of energy levels including spin splitting, nature of wavefunctions in different gauges.

•Two-dimensional (2D) electron gas – density of states splitting into Landau levels with increasing magnetic field.

•Depopulation of Landau levels with increasing magnetic field, oscillations in resistance of electron gas – Shubnikov-de-Haas effect. Measurement of 2D electron density.

•The quantum Hall effect – observed in 2D electron gas, plateau in Hall voltage when Landau level is full, provides resistance standard accurate to 5 parts in $10^8$.

# Lecture 13




**The End!!**

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