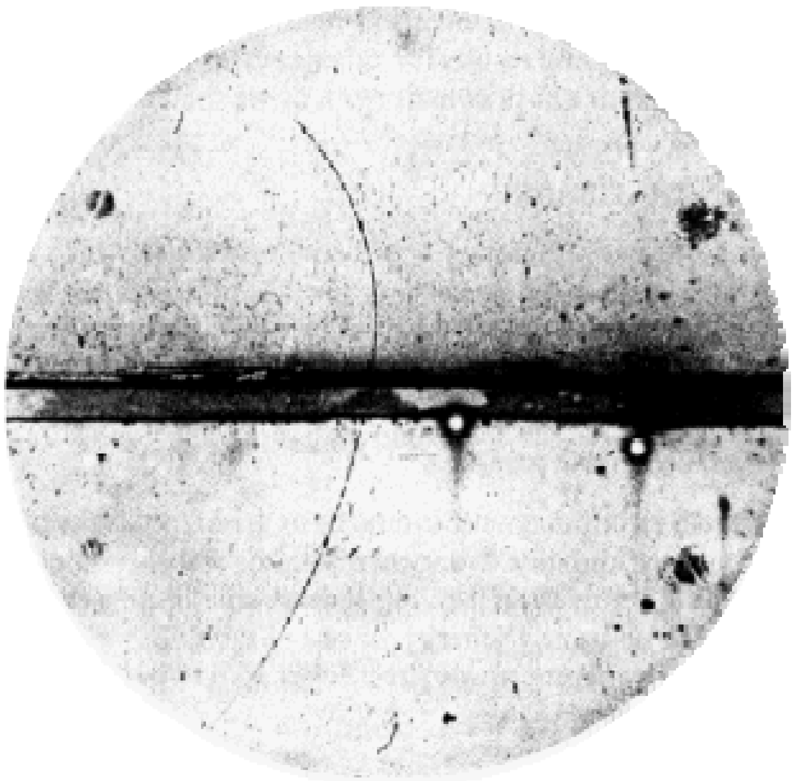
Advanced Quantum Physics

Lecture 10



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www.sp.phy.cam.ac.uk/~dar11/pdf

Section 3:Relativity and Magnetic fields

The Dirac Equation (1)

Proc. R. Soc. (Lond) **117**, 610-612 (1928)

Proc. R. Soc. (Lond) **118**, 351-361 (1928)

- 1928 Paul Dirac developed a relativistic wave equation.
- Time dependent Schrodinger equation: $i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi$
- Suppose we have a free particle so: $\mathbf{A} = \phi = 0$
- Assume that the energy operator $\hat{H}$ may be written in terms of the momentum operator $\hat{p}$ as in classical mechanics.
- From special relativity the relationship between energy and momentum is given by: $E^2 = p^2c^2 + m^2c^4$
- Hence we can suggest a form for: $\hat{H} = \sqrt{p^2c^2 + m^2c^4}$
- And the following wave equation: $i\hbar\frac{\partial\psi}{\partial t} = \sqrt{\left[\hat{p}^2c^2 + m^2c^4\right]}\psi$
- To preserve Lorentz invariance, positional coordinates and time must appear in a similar way in any theory.
- Since the momentum operators are: $\hat{p}_x = -i\hbar\frac{\partial}{\partial x}$ etc. we can write:

$$i\hbar\frac{\partial\psi}{\partial t} = \left[c\alpha_1\hat{p}_x + c\alpha_2\hat{p}_y + c\alpha_3\hat{p}_z + \beta mc^2\right]\psi$$

The Dirac equation

where α_i and β are dimensionless parameters.

The Dirac Equation (2)

•So we have two wave equations:

$$\begin{cases} i\hbar \dfrac{\partial \psi}{\partial t} = \sqrt{\left[\hat{p}^2 c^2 + m^2 c^4 \right]}\,\psi \\ i\hbar \dfrac{\partial \psi}{\partial t} = \left[c\alpha_1 \hat{p}_x + c\alpha_2 \hat{p}_y + c\alpha_3 \hat{p}_z + \beta mc^2 \right] \psi \end{cases}$$

•For consistency: $\left[c\alpha_1 \hat{p}_x + c\alpha_2 \hat{p}_y + c\alpha_3 \hat{p}_z + \beta mc^2 \right]^2 = \hat{p}^2 c^2 + m^2 c^4$

•If we multiply out the LHS and equate corresponding terms:

$$\alpha_1^2 = \alpha_2^2 = \alpha_3^2 = 1, \;\; \alpha_1\alpha_2 + \alpha_2\alpha_1 = \alpha_2\alpha_3 + \alpha_3\alpha_2 = \alpha_3\alpha_1 + \alpha_1\alpha_3 = 0$$

$$\alpha_1\beta + \beta\alpha_1 = \alpha_2\beta + \beta\alpha_2 = \alpha_3\beta + \beta\alpha_3 = 0, \;\; \beta^2 = 1$$

•Dirac showed that the simplest possible form of α_i and β was:

$$\alpha_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \;\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

•Note the Pauli spin and identity matrices within these 4x4 matrices.

$$\hat{\sigma}_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \hat{\sigma}_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \hat{\sigma}_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \hat{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The Dirac Equation (3)

- So we can rewrite the Dirac equation in the form:

$$i\hbar\frac{\partial\psi}{\partial t} = \left[c\alpha_1\hat{p}_x + c\alpha_2\hat{p}_y + c\alpha_3\hat{p}_z + \beta mc^2\right]\psi = \left[c\boldsymbol{\alpha}\cdot\hat{\mathbf{p}} + \beta mc^2\right]\psi$$

where

$$\alpha_1 = \begin{bmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{bmatrix}, \beta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$$

- The Dirac equation is effectively four PDEs linking different components of the vector:

$$\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} = \begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix}$$

- It can be written in terms of ψ_+ and ψ_- which are each two-component vectors and form two coupled equations with $\boldsymbol{\sigma} = \left(\sigma_x, \sigma_y, \sigma_z\right)$:

$$\left(\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\right)c\psi_- + mc^2\psi_+ = i\hbar\frac{\partial\psi_+}{\partial t}, \qquad \left(\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\right)c\psi_+ - mc^2\psi_- = i\hbar\frac{\partial\psi_-}{\partial t}$$

- Using $\psi = \chi\exp\left(-iEt/\hbar\right)$ we can separate out the time dependence giving:

$$\left(\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\right)c\chi_- + mc^2\chi_+ = E\chi_+, \qquad \left(\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\right)c\chi_+ - mc^2\chi_- = E\chi_-$$

The Dirac Equation (4)

- From last slide: $(\boldsymbol{\sigma}\cdot\hat{\mathbf{p}})c\chi_- + mc^2\chi_+ = E\chi_+, \qquad (\boldsymbol{\sigma}\cdot\hat{\mathbf{p}})c\chi_+ - mc^2\chi_- = E\chi_-$
- Eliminating χ_- we obtain: $(\boldsymbol{\sigma}\cdot\hat{\mathbf{p}})^2 c^2\chi_+ = (E - mc^2)(E + mc^2)\chi_+$
- Expanding the first term using $[\hat{p}_x, \sigma_x] = 0, [\hat{p}_x, \hat{p}_y] = 0$ etc. we get:

$$\begin{bmatrix} \sigma_x^2\hat{p}_x^2 + (\sigma_x\sigma_y + \sigma_y\sigma_x)\hat{p}_x\hat{p}_y \\ +\sigma_y^2\hat{p}_y^2 + (\sigma_y\sigma_z + \sigma_z\sigma_y)\hat{p}_y\hat{p}_z \\ +\sigma_z^2\hat{p}_z^2 + (\sigma_z\sigma_x + \sigma_x\sigma_z)\hat{p}_z\hat{p}_x \end{bmatrix} c^2\chi_+ = (E^2 - m^2c^4)\chi_+$$

- The Pauli matrices have the properties: $\begin{cases} \sigma_x\sigma_y = -\sigma_y\sigma_x \quad \text{etc.} \\ \sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I \end{cases}$
- And so $\hat{p}^2c^2\chi_+ = (E^2 - m^2c^4)\chi_+$
- This is solved (where $\hat{\mathbf{p}} = -i\hbar\nabla$ and χ_1, χ_2 are constants) by;

$$\chi_+ = \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} \exp(i\mathbf{k}\cdot\mathbf{r}) \quad \text{with} \quad E^2 = \hbar^2c^2k^2 + m^2c^4$$

implying $E^2 = p^2c^2 + m^2c^4$ if the de Broglie relation $p = \hbar k$ is obeyed.

The Dirac Equation (5)

- The last slide: $(\boldsymbol{\sigma}\cdot\hat{\mathbf{p}})c\chi_+ - mc^2\chi_- = E\chi_-, \qquad (\boldsymbol{\sigma}\cdot\hat{\mathbf{p}})c\chi_- + mc^2\chi_+ = E\chi_+$
- Which imply: $\chi_- = \frac{c}{E+mc^2}(\boldsymbol{\sigma}\cdot\hat{\mathbf{p}})\chi_+, \qquad \chi_+ = \frac{c}{E-mc^2}(\boldsymbol{\sigma}\cdot\hat{\mathbf{p}})\chi_-$
- There are two different possible values for the energy:

$$E_+ = +\sqrt{p^2c^2+m^2c^4}, \qquad E_- = -\sqrt{p^2c^2+m^2c^4}$$

- If we assume that $E = E_+$ and take the non-relativistic limit: $pc \ll mc^2 + E$ then $\chi_+ \gg \chi_-$ and the overall time-independent wavefunction χ is just:

$$\chi = \begin{bmatrix}\chi_+\\ \chi_-\end{bmatrix} = \begin{bmatrix}\chi_1\\ \chi_2\\ 0\\ 0\end{bmatrix}\exp(i\mathbf{k}\cdot\mathbf{r})$$

- χ_+ has the form of the product of a plane wave and a two component vector – this is appropriate for a particle with $S=\frac{1}{2}$ and χ_1, χ_2 determining the spin direction.

- So....spin is emerging naturally from the solutions of Dirac's equation at low energies.

The Dirac Equation (6)

- From the last slide: $\chi_- = \frac{c}{E+mc^2}(\boldsymbol{\sigma}\cdot\hat{\mathbf{p}})\chi_+, \qquad \chi_+ = \frac{c}{E-mc^2}(\boldsymbol{\sigma}\cdot\hat{\mathbf{p}})\chi_-$

- If we take the negative energy solution $E = E_- = -\sqrt{p^2c^2+m^2c^4}$ in the non-relativistic limit $pc \ll mc^2 + E$ then $\chi_- \gg \chi_+$ and the overall wavefunction is given by:

$$\chi = \begin{bmatrix} \chi_+ \\ \chi_- \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \chi_3 \\ \chi_4 \end{bmatrix} \exp(i\mathbf{k}\cdot\mathbf{r})$$

- So... this is a particle with spin $S = \frac{1}{2}$ and negative energy!!

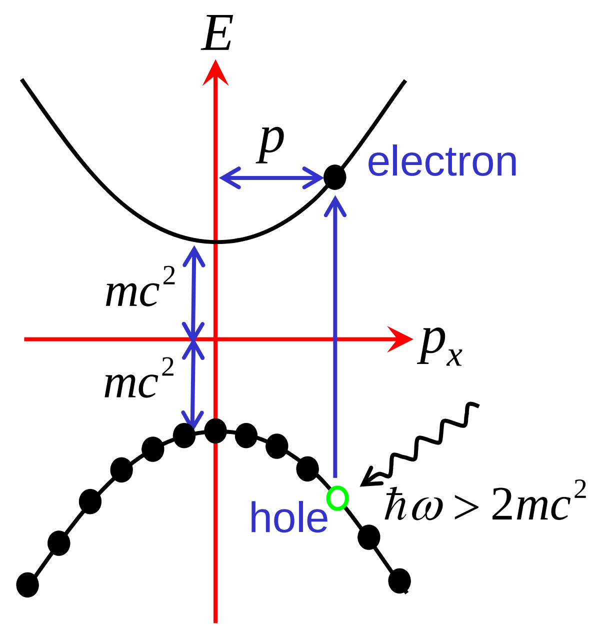
Antiparticles (1)

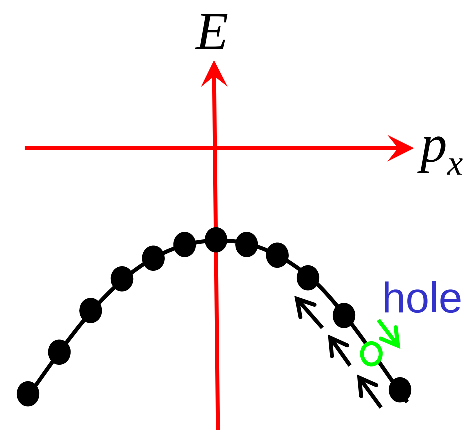
- The classical expression $E^2 = p^2c^2 + m^2c^4$ gives no restriction on the sign of E but in practice these solutions are rejected as being unphysical – there is also no mechanism for it.
- Quantum mechanical transitions could allow these states to be reached.
- For example: an interaction with an electromagnetic perturbation could cause an electron with energy greater than its rest mass mc^2 to make a radiative transition to a state of negative energy less than $-mc^2$.
- In 1930 Dirac * proposed a way out of this by suggesting that all the negative energy states in the universe are full. Because of the Pauli exclusion principle only one electron can occupy each state so transitions from positive energy to negative energy states are forbidden.
- Hence it is possible for radiation with energy $E > 2mc^2$ to interact with a negative energy electron causing it to make a transition to a positive energy state leaving behind a 'hole'.
- Initially he thought these 'holes' corresponded to protons – but the mass difference was a problem....

*Proc. Royal Soc. (Lond) A, 126, 360-5 (1930)

Antiparticles (2)

- We can represent this process in a diagram of energy versus momentum:
- An electron is moved from a negative energy state to a positive energy state creating a vacancy or 'hole'.
- The total energy of the negative states has increased by $\geq mc^2$ and their net momentum is now $-p$.
- Now apply an electric field in the $+x$ direction.
- The negatively charged electrons accelerate in the direction opposite to the field their momentum changing in the $-p_x$ direction.
- The hole moves in the $+p_x$ direction as one would expect for a positively charged particle with the same mass as an electron.
- Photon has created an electron-positron pair - opposite annihilation process gives out photon(s).





Note: Similar ideas of electrons and holes are used extensively in solid state physics.

Antiparticles (3)

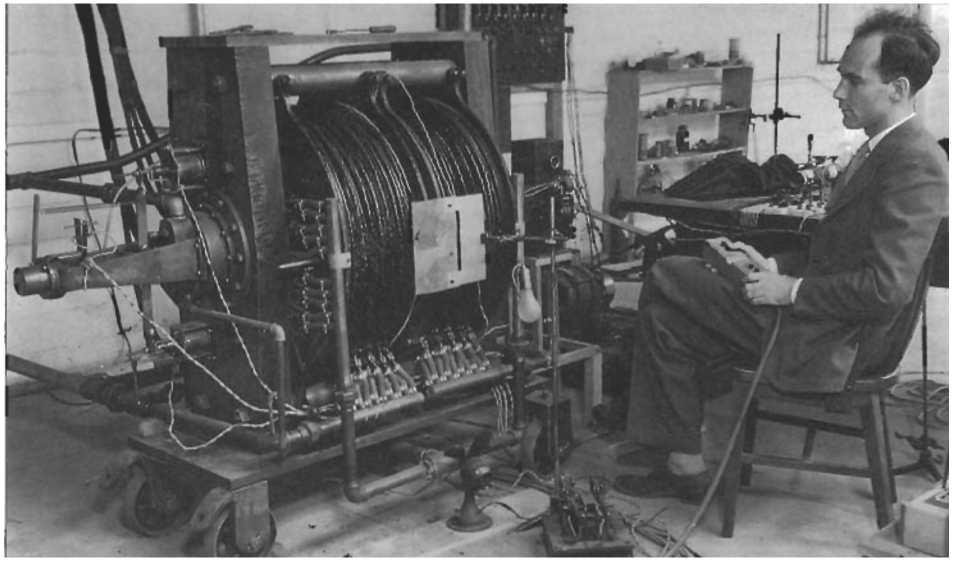
- There are some problems with Dirac's model:
- The sea of occupied states has no observable properties until a hole is created – so an infinite set of particles has no mass or charge!!!
- No symmetry between electrons and positrons – electrons could be holes in a sea of positrons.
- Later quantum field theories got around this – particle and antiparticle pairs are excited states of 'Dirac field' the ground state being the vacuum.
- Dirac also suggested the existence of the anti-proton and even went as far as to suggest that other solar systems might be made up entirely of anti-particles.
- Dirac received the Nobel prize in 1933 after the experimental observation of the positron. His Nobel lecture 'Theory of electrons and positrons' is on the course web site.

The experimental observation of the positron

•Following the observation of tracks of cosmic rays in 1927 by Skolbeltzyn, in 1930 at Caltech Robert Millikan and Carl Anderson started to design and build a cloud chamber to study these particles.

•The cloud chamber (invented by CTR Wilson at the Cavendish – look at the Museum) works by the sudden expansion of air containing water vapour forming a mist. Droplets are formed by the passage of charged particles and a photograph is taken of the resulting track.

•Anderson designed and had built a cloud chamber of dimensions 17x17x3cm which was incorporated into an electromagnet capable of supplying a field up to 24,000 gauss (2.4T), ten times higher than had previously been available. The magnet coils were made of copper tubing cooled by tap water and it consumed up to 600kW in power.



Phys Rev **43**, 491 (1933)

The experimental observation of the positron (2)

•On August 2 1932 Carl Anderson at Caltech was using his cloud chamber to photograph cosmic ray tracks.

•A magnetic field of 15,000 Gauss (1.5T) was used.

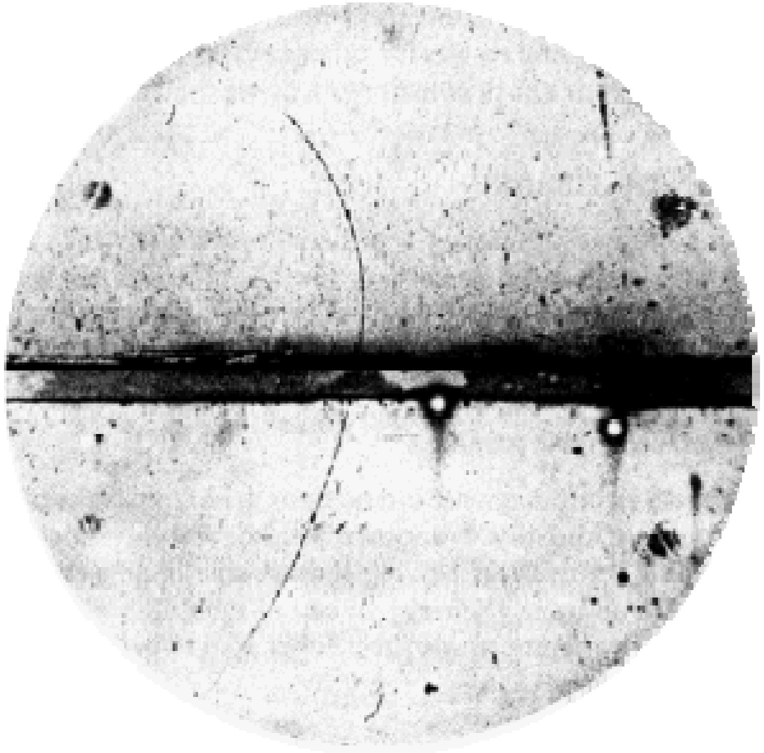
•A particle moved from bottom of diagram through a lead plate had opposite curvature to an electron – positive charge!!

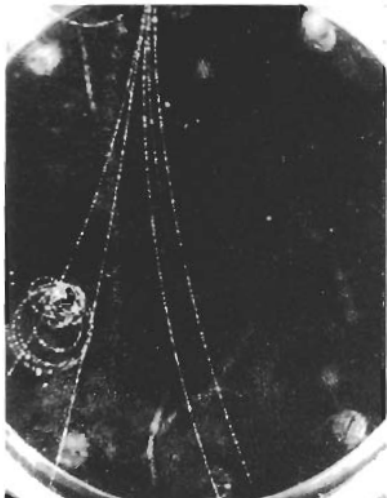
• Curvature of path increased after going through the plate.

•Overall 15 positive particles in 1300 photos were found.

•Particle track of similar curvature could be made by a 300keV proton but... it would only have a range of 5mm not the 5cm observed.

•Anderson won the Nobel prize in 1936





A cosmic ray interacting with the cloud chamber wall produces electrons and positrons which curve in different directions in the magnetic field

Lecture 10 - Summary

- The Dirac equation – a relativistic wave equation consistent with the Schrodinger equation and the relativistic energy-momentum relation.

$$i\hbar\frac{\partial\psi}{\partial t} = \left[c\alpha_1\hat{p}_x + c\alpha_2\hat{p}_y + c\alpha_3\hat{p}_z + \beta mc^2\right]\psi$$

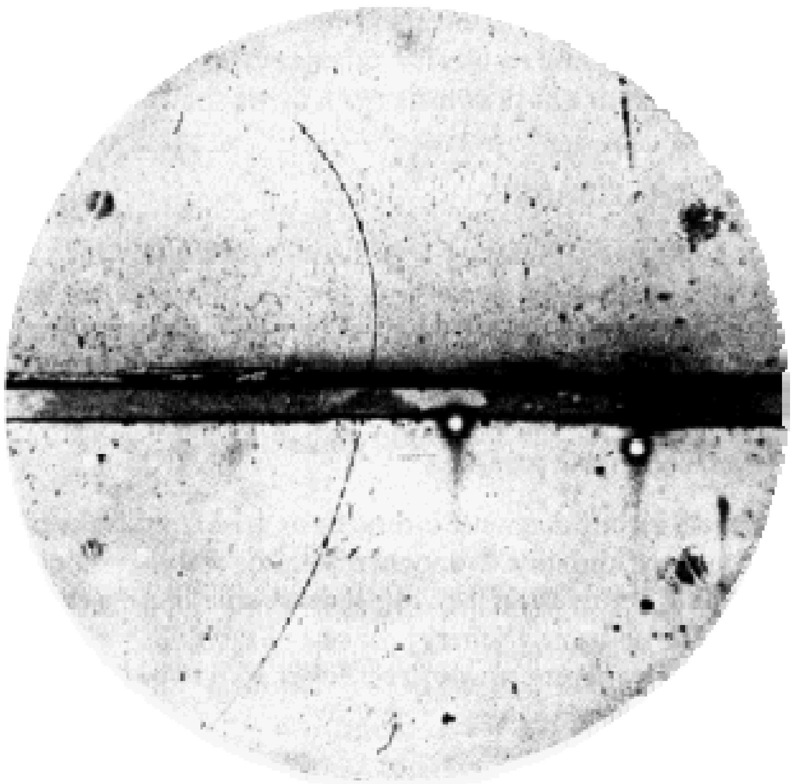
- The Dirac equation incorporates 4x4 matrices $[\alpha_1, \alpha_2, \alpha_3, \beta]$ and has 4 component vectors ψ as solutions – effectively it consists of 4 coupled partial differential equations.

- Solutions imply negative and positive energies and a spin degree of freedom.

- Prediction of anti-particles, concept of *holes*.

- Experimental observation of the positron (1932) – cosmic ray tracks observed using a cloud chamber in a magnetic field.

Lecture 10



The End!!

(www.sp.phy.cam.ac.uk/~dar11/pdf)